An integrable system on the moduli space of rational functions and its variants

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Abstract

We study several integrable Hamiltonian systems on the moduli spaces of meromorphic functions on Riemann surfaces (the Riemann sphere, a cylinder and a torus). The action–angle variables and the separated variables (in Sklyanin's sense) are related via a canonical transformation, the generating function of which is the Abelian type integral of the Seiberg–Witten differential over the spectral curve.
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1. Introduction

Moser's work [1] on the open finite Toda lattice will be presumably the earliest attempt in the literature to relate a moduli space of rational functions to an integrable system. The idea is to construct a rational function

$$f(\lambda) = \sum_{j=1}^{N} \frac{\rho_j}{\lambda - \alpha_j}$$

from the *L*-matrix of the Toda lattice, which can be reproduced from $f(\lambda)$ by a continued fraction. Moser discovered that the dynamics of the Toda lattice is linearized in the new

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variables ρ_j and α_j (i.e., moduli of the rational function), so that an explicit formula of solutions of the Toda lattice can be obtained by inverting the map $L \to f(\lambda)$. The mechanism of linearization was elucidated (and generalized) by Kostant [2] in the language of representations of Lie groups.

It was soon noticed that Moser's method resembles mathematical techniques of the system control theory in engineering. An analog of Moser's rational function emerges therein as a fundamental quantity (the rational transfer function) that characterizes the input–output relation of a linear control system. Following Moser's construction, Krishnaprasad [3] introduced dynamical flows on the moduli space of such rational functions, and solved some geometrical problems raised by Brockett [4]. Krishnaprasad's work was further refined by Nakamura [5], who thus obtained a few variants of the open finite Toda lattice.

Another notable example where a moduli space of rational functions plays a role, is the monopole moduli space of the SU(2) Yang–Mills theory. According to Donaldson's result [6], the monopole moduli space is isomorphic to a space of rational functions (though with complex coefficients, as opposed to Moser's case). It is not difficult to infer the existence of an integrable system on this moduli space from Donaldson's work (and the subsequent progress in the book of Atiyah and Hitchin [7]). Firstly, although Donaldson mentioned nothing, Donaldson's construction of the rational function is actually very similar to the rational transfer function in the system control theory. Secondly, Atiyah and Hitchin [7] introduced a symplectic structure on Donaldson's space of rational functions, which can be the first step towards the construction of an integrable system. From the first point of view, Nakamura [8] extended his previous work [5] to this space of complex rational functions. The second point of view was sought by Faybusovich and Gekhtman [9], who recently reported the existence of a multi-Hamiltonian structure therein [10].

In this paper, we take up this relatively well understood integrable system from two new aspects:

1. *Action–angle variables of Seiberg–Witten type*. Needless to say, the existence of action–angle variables is the most fundamental aspect of integrable systems. The so-called Seiberg–Witten integrable systems [11–13] are characterized by a special set of action–angle variables that are associated with a special differential dS (the Seiberg–Witten differential) on the spectral curve. We shall show that the integrable system on the moduli space of rational functions has a similar property.
2. *Separation of variables (SoV)*. Another remarkable aspect of many finite-dimensional integrable systems is SoV. In the modern version of SoV due to Sklyanin [14], an integrable system is mapped to a dynamical system of a finite number of points on the spectral curve, and the coordinates of these moving points are nothing but separated variables. (A prototype of this interpretation of SoV can be found in Moser's work on classical integrable systems [15] and its generalization by the Montreal group [16].) The integrable system on the moduli space of rational functions turns out to have a similar set of separated variables.

The role of the spectral curve is now played by a *rational curve* of the form

$$C = \{(\lambda, z) | z = A(\lambda)\},$$

where $A(\lambda)$ is the denominator of the expression

$$f(\lambda) = \frac{B(\lambda)}{A(\lambda)}$$

of $f(\lambda)$ as the quotient of two polynomials. The separated variables are given by the zeros λ_j of $B(\lambda)$ and the values $z_j = A(\lambda_j)$ of $A(\lambda)$ at these zeros. In the context of Seiberg–Witten integrable systems, this kind of rational spectral curves emerge in the weak coupling limit of the supersymmetric gauge theory [17,18].

We construct two variants of this integrable system on the basis of these observations. The new integrable systems live on a moduli space of trigonometric or elliptic functions, and inherit most of the properties of the rational case. We show that these integrable systems can be formulated in a unified way, which will be useful for higher genus generalization.

Further generalization can be sought in the perspective of complex surfaces with symplectic or Poisson structure [19–21]. All the three (rational, trigonometric and elliptic) cases are associated with a complex surface X with rational fibration over a (rational or elliptic) curve Σ. The curve C is embedded in X. A natural idea of generalization is to replace the rational fibration by elliptic fibration. A few examples of that kind of integrable systems have been constructed [22]. We re-examine these examples in the present context.

A few more comments on the rational case are in order. Firstly, Morosi and Tondo [23] considered a very similar example of SoV. Their example is picked out from Calogero's many-body systems describing motion of zeros (or poles) of solutions of a partial differential equation [24]. This example, too, can be reformulated as an integrable system on the moduli space of rational functions except that the degree of the denominator $B(\lambda)$ is different from our case. Secondly, Krichever and Vaninsky [25] developed an algebro-geometric approach to the finite open Toda lattice using Baker–Akhiezer functions on a singular rational curve. The Seiberg–Witten structure and separated variables can be reproduced from their results as well.

This paper is organized as follows. In Section 2, we show an outline of our construction. Section 3 reviews the case of rational functions from the new points of view. Integrable systems on a space of trigonometric and elliptic functions are constructed in Sections 4 and 5. Section 6 seeks generalizations in the context of symplectic surfaces. We show our conclusion in Section 7.

2. Outline of construction

Our strategy of constructing the integrable systems is summarized in the following way. We do not consider global nature of the phase space and restricted ourselves to generic situation.

1. The phase space is the moduli space of meromorphic functions of certain class on a fixed Riemann surface Σ, which is either a sphere, a cylinder or a torus. We factorize each meromorphic function as $B(\lambda)/A(\lambda)$, where $A(\lambda)$ and $B(\lambda)$ are holomorphic functions (or holomorphic sections of a line bundle) on Σ.

2. Let $\{\alpha_j\}$ be the set of poles of the meromorphic function $B(\lambda)/A(\lambda)$, i.e., zeros of $A(\lambda)$. We have a canonical coordinate system on the phase space $(\alpha_j, \psi_j)_{j=1,\ldots,N}$. Hence the canonical 2-form Ω of the phase space has the form

$$\Omega = \sum_{j=1}^{N} d\alpha_j \wedge d\psi_j. \tag{1}$$

3. The Hamiltonians of the system $u_n(\alpha)$ $(n = 1, \ldots, N)$ are given. Not containing the variables ψ_j, they mutually commute; $\{u_n, u_m\} = 0$. By coordinate transformation, we can rewrite Ω in the form

$$\Omega = \sum_{n=1}^{N} du_n \wedge d\phi_n. \tag{2}$$

   Namely, ϕ_n is the angle variable corresponding to the action variable u_n. In this sense, the system is trivially solved.

4. There exists another set of canonical variables $(\lambda_j, \mu_j)_{j=1,\ldots,N}$ for which Ω can be written as

$$\Omega = \sum_{j=1}^{N} d\mu_j \wedge d\lambda_j, \tag{3}$$

   where $\{\lambda_j\}$ is the set of zeros of $B(\lambda)/A(\lambda)$, i.e., zeros of $B(\lambda)$. Each pair $(\lambda_j, z_j = e^{\mu_j})$ $(j = 1, \ldots, N)$ satisfies a relation,

$$z_j = A(\lambda_j; u_1, \ldots, u_N). \tag{4}$$

5. The generating function $S(\lambda_1, \ldots, \lambda_N; u_1, \ldots, u_N)$ of the canonical transformation

$$(\lambda_j, \mu_j)_{j=1,\ldots,N} \mapsto (u_n, \phi_n)_{n=1,\ldots,N} \tag{5}$$

   is defined by the integral

$$S(\lambda_1, \ldots, \lambda_N; u_1, \ldots, u_N) = \sum_{i=1}^{N} \int^{\lambda_i} \log A(\lambda; u_1, \ldots, u_N) \, d\lambda. \tag{6}$$

   Hence the variables μ_j and ϕ_n are expressed as

$$\mu_j = \frac{\partial S}{\partial \lambda_j}, \qquad \phi_n = \frac{\partial S}{\partial u_n} = \sum_{i=1}^{N} \int^{\lambda_i} \frac{\partial A}{\partial u_n}(\lambda; u_1, \ldots, u_N) \frac{d\lambda}{A(\lambda; u_1, \ldots, u_N)}. \tag{7}$$

6. By the interpolation formulae we can solve Eq. (4) and express the Hamiltonian u_n in terms of the variables $(\lambda_j, z_j)_{j=1,\ldots,N}$; $u_n = u_n(\lambda_1, \ldots, \lambda_n; z_1, \ldots, z_N)$. Thus we obtain a non-trivial integrable (or integrated) system.

We have three canonical coordinate systems, (α_j, ψ_j), (u_n, ϕ_n) and (λ_j, μ_j). The first two are of action–angle type while the third are *separated variables*. In fact, (4) may be interpreted as the equation of the common level set $u_n = u_n(z_1, \ldots, z_N; \lambda_1, \ldots, \lambda_n)$ of

the Hamiltonians expressed in separated variables (see [14]). Note that, in general, if one solves the equations of the common level set by the implicit function theorem, the result would be

$$z_j = \Phi_j(\lambda_1, \ldots, \lambda_N; u_1, \ldots, u_N).$$

We emphasize that the right-hand side of (4) contains only λ_j among all $\lambda_1, \ldots, \lambda_N$. This is the key point in the SoV method.

Usually, the SoV method requires a relation of the form $\Psi_j(z_j, \lambda_j; u_1, \ldots, u_N) = 0$ for each j. Our assumption (4) says more than that; N points (λ_j, z_j) lie on one and the same curve $C_u := \{(\lambda, z) | z = A(\lambda; u_1, \ldots, u_N)\}$ parameterized by $u_1, \ldots, u_N$.

In this respect, the definition (6) of the generating function of the canonical transformation is the Abelian type integral of the *Seiberg–Witten differential* $\log A(\lambda; u) \, d\lambda$ on the curve C_u. We can also interpret the coordinate transformation $(\alpha_1, \ldots, \alpha_N) \mapsto (u_1, \ldots, u_N)$ as the inverse map of the *period integral*

$$\alpha_j = \oint_{a_j} \lambda \, d_\lambda \log A(\lambda; u_1, \ldots, u_N), \tag{8}$$

where a_j is the cycle on the curve C_u encircling a zero of $A(\lambda; u)$ on the λ-plane. The interpolation formulae $u_n = u_n(\lambda; z)$ determines the moduli of the curve C_u from the points lying on it.

3. Rational case

Let us consider the moduli space of rational functions $B(\lambda)/A(\lambda)$, where A and B are polynomials of the form

$$A(\lambda) = \lambda^N + u_1\lambda^{N-1} + u_2\lambda^{N-2} + \cdots + u_N = \prod_{j=1}^N (\lambda - \alpha_j), \tag{9}$$

$$B(\lambda) = \rho \prod_{k=1}^{N-1} (\lambda - \lambda_k). \tag{10}$$

For simplicity, we assume that all roots of A and B are distinct. Since $B(\lambda)/A(\lambda)$ has a partial fraction expansion

$$\frac{B(\lambda)}{A(\lambda)} = \sum_{j=1}^N \frac{\rho_j}{\lambda - \alpha_j}, \quad \rho = \sum_{j=1}^N \rho_j, \quad \rho_j = \frac{B(\alpha_j)}{A'(\alpha_j)}, \tag{11}$$

there are two coordinates of this $2N$-dimensional moduli space, $(\alpha_1, \ldots, \alpha_N, \rho, \lambda_1, \ldots, \lambda_{N-1})$ and $(\rho_1, \ldots, \rho_N, \alpha_1, \ldots, \alpha_N)$.

Following Atiyah and Hitchin [7], we introduce the symplectic form

$$\Omega = \sum_{j=1}^N d \log B(\alpha_j) \wedge d\alpha_j \tag{12}$$

on this space. Since each u_n is the elementary symmetric function of α_j, they commute with each other; $\{u_m, u_n\} = 0$.

By an easy residue calculus, we can rewrite this form as follows:

$$\Omega = \sum_{j=1}^{N} \text{Res}_{\lambda=\alpha_j} \left(\frac{\delta A(\lambda)}{A(\lambda)} \wedge \frac{\delta B(\lambda)}{B(\lambda)} \wedge d\lambda \right), \tag{13}$$

where δ is the exterior derivation with respect to the moduli coordinates, i.e., $\delta = d - d\lambda(\partial/\partial\lambda)$. The 3-form inside the bracket is rational on the λ-sphere with poles at $\lambda = \alpha_j$ $(j = 1, \ldots, N)$, $\lambda = \lambda_k$ $(k = 1, \ldots, N-1)$ and $\lambda = \infty$. The residues at $\lambda = \lambda_k$ and $\lambda = \infty$ are

$$\text{Res}_{\lambda=\lambda_k} \left(\frac{\delta A(\lambda)}{A(\lambda)} \wedge \frac{\delta B(\lambda)}{B(\lambda)} \wedge d\lambda \right) = -d \log A(\lambda_k) \wedge d\lambda_k,$$

$$\text{Res}_{\lambda=\infty} \left(\frac{\delta A(\lambda)}{A(\lambda)} \wedge \frac{\delta B(\lambda)}{B(\lambda)} \wedge d\lambda \right) = du_1 \wedge d \log \rho.$$

By the residue theorem, we have another expression of the form Ω from (13),

$$\Omega = \sum_{k=1}^{N-1} d \log A(\lambda_k) \wedge d\lambda_k - du_1 \wedge d \log \rho. \tag{14}$$

Let us consider the restricted moduli space $\{\rho = 1, u_1 = 0\}$ (the "reduced monopole space"; see [7, p. 19]) and the restriction of Ω

$$\Omega|_{\rho=1, u_1=0} = \sum_{k=1}^{N-1} d \log A(\lambda_k) \wedge d\lambda_k. \tag{15}$$

Substituting the explicit expression (9), we have

$$\Omega|_{\rho=1, u_1=0} = \sum_{k=1}^{N-1} \sum_{j=2}^{N} \frac{\lambda_k^{N-j} du_j \wedge d\lambda_k}{A(\lambda_k)} = \sum_{j=2}^{N} du_j \wedge \left(\sum_{k=1}^{N-1} \frac{\lambda_k^{N-j}}{A(\lambda_k)} d\lambda_k \right). \tag{16}$$

We introduce the generating function of the canonical transformation by

$$S(\lambda_1, \ldots, \lambda_{N-1}; u_2, \ldots, u_N) := \sum_{k=1}^{N-1} \int^{\lambda_k} \log A(\lambda) \, d\lambda. \tag{17}$$

The form $dS = \log A(\lambda) \, d\lambda$ is a Seiberg–Witten differential on the curve $C_u = \{(\lambda, z) | z = A(\lambda)\}$ and the right-hand side of (17) is an Abelian type integral of dS corresponding to the divisor $\sum_{k=1}^{N-1} (\lambda_k, z_k)$ on C_u.

The variable $\mu_k = \log z_k$ and the "angle variable" ϕ_n corresponding to u_n is expressed as

$$\mu_k = \frac{\partial S}{\partial \lambda_k}, \qquad \phi_n = \frac{\partial S}{\partial u_n} = \sum_{k=1}^{N-1} \int^{\lambda_k} \frac{\lambda^{N-j}}{A(\lambda)} \, d\lambda, \tag{18}$$

and we can rewrite (16) as

$$\Omega|_{\rho=1,u_1=0} = \sum_{j=2}^{N} \mathrm{d}u_j \wedge \mathrm{d}\phi_n. \tag{19}$$

The Hamiltonians u_n ($n = 2, \ldots, N$) are explicitly written down in terms of the coordinate system ($\lambda_1, \ldots, \lambda_{N-1}, z_1, \ldots, z_{N-1}$), where $z_k = A(\lambda_k)$. This is a simple application of the Lagrange interpolation formula

$$\frac{A(\lambda) - \lambda^N}{B(\lambda)} = \sum_{k=1}^{N-1} \frac{A(\lambda_k) - \lambda_k^N}{B'(\lambda_k)} \frac{1}{\lambda - \lambda_k} = \sum_{k=1}^{N-1} \frac{z_k - \lambda_k^N}{B'(\lambda_k)} \frac{1}{\lambda - \lambda_k}. \tag{20}$$

(Note that the left-hand side has poles at $\lambda = \lambda_k$ by (10).) For example, u_2 is read off directly from the residue of (20) at $\lambda = \infty$

$$u_2 = \sum_{k=1}^{N-1} \frac{z_k - \lambda_k^N}{B'(\lambda_k)} = \sum_{k=1}^{N-1} \frac{z_k - \lambda_k^N}{\prod_{l \neq k}(\lambda_k - \lambda_l)}. \tag{21}$$

If we denote the coefficients of B by v_j,

$$B(\lambda) = \lambda^{N-1} + \sum_{j=1}^{N} v_j \lambda^{N-j-1}, \tag{22}$$

other u_j's can be expressed as

$$u_{j+1} = -\sum_{k=1}^{N-1} \frac{z_k - \lambda_k^N}{B'(\lambda_k)} \frac{\partial v_j}{\partial \lambda_k} \tag{23}$$

for $j = 1, \ldots, N - 1$. (This expression recovers (21), since $v_1 = -\sum_{k=1}^{N-1} \lambda_k$.)

4. Trigonometric (hyperbolic) case

In the trigonometric case, we consider the moduli space of meromorphic functions $B(\lambda)/A(\lambda)$, where A and B are trigonometric polynomials

$$A(\lambda) = \prod_{j=1}^{N} \sinh(\lambda - \alpha_j), \tag{24}$$

$$B(\lambda) = \prod_{k=1}^{N} \sinh(\lambda - \lambda_k). \tag{25}$$

For simplicity, we assume that λ_k's and α_j's are distinct. Since $B(\lambda)/A(\lambda)$ has a period $\pi\mathrm{i}$, α_j and λ_k are regarded as points on the cylinder, $\mathbb{C}/\pi\mathrm{i}\mathbb{Z}$.

The symplectic form

$$\Omega = \sum_{j=1}^{N} \mathrm{d} \log B(\alpha_j) \wedge \mathrm{d}\alpha_j \tag{26}$$

is a trigonometric analog of the form (12). The expression (13) holds also in this case. To obtain the formula like (14) or (15), we can apply the residue formula which expresses the sum of the residues as the contour integral, but the following argument is simpler: the logarithmic derivative of $\sinh \lambda$ is the hyperbolic cotangent, $\coth \lambda$, which is an odd function, $\coth(-\lambda) = -\coth \lambda$. Therefore, it follows from the definition (26) that:

$$\Omega = -\sum_{j,k=1}^{N} \coth(\alpha_j - \lambda_k) \, \mathrm{d}\lambda_k \wedge \mathrm{d}\alpha_j = -\sum_{j,k=1}^{N} \coth(\lambda_k - \alpha_j) \, \mathrm{d}\alpha_j \wedge \mathrm{d}\lambda_k$$

$$= \sum_{k=1}^{N} \mathrm{d} \log A(\lambda_k) \wedge \mathrm{d}\lambda_k. \tag{27}$$

As in the rational case the coefficients of $A(\lambda)$ become action variables. Let us denote e^λ by x and expand $A(\lambda)$ as

$$A(\lambda) = \frac{1}{2^N}(u_0 x^N - u_1 x^{N-2} + \cdots + (-1)^{N-1} u_{N-1} x^{2-N} + (-1)^N u_N x^{-N}). \tag{28}$$

Each coefficient u_n is expressed explicitly as

$$u_n = \sum_{\substack{I_1 \sqcup I_2 = \{1,\ldots,N\} \\ |I_1| = n}} \prod_{i \in I_1} \mathrm{e}^{\alpha_i} \prod_{j \in I_2} \mathrm{e}^{-\alpha_j}. \tag{29}$$

In particular,

$$u_0 = \exp\left(-\sum_{j=1}^{N} \alpha_j\right), \qquad u_N = u_0^{-1}. \tag{30}$$

Since u_n's are functions of α_j, they commute with each other with respect to the Poisson bracket defined by the form Ω, (26).

Taking the formula (30) into account, we have

$$\mathrm{d} \log A(\lambda_k) \wedge \mathrm{d}\lambda_k$$

$$= \frac{1}{2^N}\left(\sum_{n=1}^{N-1} \mathrm{d}u_n \wedge \frac{(-1)^n x_k^{N-2n}}{A(\lambda_k)} \, \mathrm{d}\lambda_k + \frac{\mathrm{d}u_0}{u_0} \wedge \frac{u_0 x_k^N - (-1)^N u_0^{-1} x_k^{-N}}{A(\lambda_k)} \, \mathrm{d}\lambda_k\right), \tag{31}$$

where $x_k = \mathrm{e}^{\lambda_k}$. Summing them up, we have from (27) that

$$\Omega = \sum_{n=1}^{N-1} \mathrm{d}u_n \wedge \mathrm{d}\phi_n + \frac{\mathrm{d}u_0}{u_0} \wedge \mathrm{d}\phi_0, \tag{32}$$

where ϕ_n's are angle variables defined by

$$\phi_n = \frac{1}{2^N} \sum_{k=1}^{N} \int^{\lambda_k} \frac{(-1)^n \, e^{(N-2n)\lambda}}{A(\lambda)} \, d\lambda \tag{33}$$

for $n = 1, \ldots, N-1$ and

$$\phi_0 = \frac{1}{2^N} \sum_{k=1}^{N} \int^{\lambda_k} \frac{u_0 \, e^{N\lambda} - (-1)^N u_0^{-1} \, e^{-N\lambda}}{A(\lambda)} \, d\lambda. \tag{34}$$

The generating function of the canonical transformation $(\lambda_k, \mu_k = \log A(\lambda_k))_{k=1,\ldots,N} \mapsto (u_n, \phi_n)_{n=1,\ldots,N}$ is

$$S(\lambda_1, \ldots, \lambda_N; u_1, \ldots, u_N) = \sum_{k=1}^{N} \int^{\lambda_k} \log A(\lambda) \, d\lambda, \tag{35}$$

and the variable μ_k and ϕ_n are expressed as

$$\mu_k = \frac{\partial S}{\partial \lambda_k}, \qquad \phi_n = \frac{\partial S}{\partial u_n}. \tag{36}$$

The right-hand side of (35) is again the Abelian integral of the Seiberg–Witten differential $\log A(\lambda) \, d\lambda$ on the curve $C_u = \{(\lambda, z) | z = A(\lambda)\}$ parameterized by $u_1, \ldots, u_N$.

Explicit expressions of the Hamiltonians u_n ($n = 0, \ldots, N-1$) in terms of the coordinate system $(\lambda_1, \ldots, \lambda_{N-1}, z_1, \ldots, z_{N-1})$, $z_k = A(\lambda_k)$, are obtained from the interpolation formula. By comparing the periodicity, position of poles and their residue, it is easy to see that

$$\frac{A(\lambda)}{B(\lambda)} = \sum_{k=1}^{N} \frac{z_k}{B'(\lambda_k)} \coth(\lambda - \lambda_k) + c_0, \tag{37}$$

where c_0 is a constant independent of λ but dependent on λ_k and α_j. Let us expand $B(\lambda)$ as

$$B(\lambda) = \frac{1}{2^N}(v_0 x^N - v_1 x^{N-2} + \cdots + (-1)^{N-1} v_{N-1} x^{2-N} + (-1)^N v_N x^{-N}), \tag{38}$$

where

$$v_n = \sum_{\substack{I_1 \sqcup I_2 = \{1,\ldots,N\} \\ |I_1| = n}} \prod_{i \in I_1} e^{\lambda_i} \prod_{j \in I_2} e^{-\lambda_j}. \tag{39}$$

In particular, $v_0 = \exp\left(-\sum_{k=1}^{N} \lambda_k\right)$, $v_N = v_0^{-1}$. Summing up the asymptotics of the both hand sides of (37) in the limit $\mathrm{Re}\,\lambda \to \pm\infty$, we have

$$\frac{u_0}{v_0} + \frac{v_0}{u_0} = 2c_0, \quad \text{i.e.,} \quad c_0 = \frac{1}{2}\left(\frac{u_0}{v_0} + \frac{v_0}{u_0}\right) = \cosh\left(\sum_{k=1}^{N} \lambda_k - \sum_{j=1}^{N} \alpha_j\right), \tag{40}$$

because $\coth \lambda \to \pm 1$ ($\mathrm{Re}\, \lambda \to \pm\infty$). Expansion of (37) around $x = \mathrm{e}^\lambda = \infty$ gives

$$
\begin{aligned}
u_0 &- u_1 x^{-2} + \cdots + (-1)^{N-1} u_{N-1} x^{2-2N} + (-1)^N u_N x^{-2N} \\
&= (v_0 - v_1 x^{-2} + \cdots + (-1)^{N-1} v_{N-1} x^{2-2N} + (-1)^N v_N x^{-2N}) \\
&\quad \times \left(c_0 + \sum_{k=1}^N (1 + 2x^{-2}\, \mathrm{e}^{2\lambda_k} + 2x^{-4}\, \mathrm{e}^{4\lambda_k} + \cdots) \frac{z_k}{B'(\lambda_k)} \right).
\end{aligned}
\tag{41}
$$

(Note that $\coth \lambda = 1 + \sum_{n=1}^\infty 2x^{-2n}$ for $|x| > 1$.) By comparing the coefficients of x^{-2n} we obtain the expression of u_n. For example, the coefficients of x^0 is

$$
u_0 = v_0 \left(c_0 + \sum_{k=1}^N \frac{z_k}{B'(\lambda_k)} \right),
\tag{42}
$$

which is equivalent to

$$
\sum_{k=1}^N \frac{z_k}{B'(\lambda_k)} = \frac{1}{2} \left(\frac{u_0}{v_0} - \frac{v_0}{u_0} \right) = \sinh \left(\sum_{j=1}^N \lambda_j - \sum_{k=1}^N \alpha_j \right)
\tag{43}
$$

due to (40).[1] Therefore u_0 is the solution of the quadratic equation

$$
u_0^2 - 2 \left(\sum_{k=1}^N \frac{z_k}{B'(\lambda_k)} \right) v_0 u_0 - v_0^2 = 0.
\tag{44}
$$

The coefficients of x^{-2} of (41) gives the expression of u_1 as follows:

$$
u_1 = 2v_0 \sum_{k=1}^N \frac{\mathrm{e}^{2\lambda_k} z_k}{B'(\lambda_k)} - \frac{v_1}{v_0} u_0,
\tag{45}
$$

and other u_n's are determined recursively.

5. Elliptic case

In the elliptic case, we take the following elliptic polynomials $A(\lambda)$ and $B(\lambda)$:

$$
A(\lambda) = \prod_{j=1}^N \sigma(\lambda - \alpha_j),
\tag{46}
$$

$$
B(\lambda) = \prod_{k=1}^N \sigma(\lambda - \lambda_k),
\tag{47}
$$

---

[1] We can derive this relation directly from the interpolation formula (37). In fact, the left-hand side of (43) is the sum of residues of (37). The right-hand side is the result of a contour integral of $A(\lambda)/B(\lambda)$ along the rectangle with vertices R, $R + \pi\mathrm{i}$, $-R + \pi\mathrm{i}$, $-R$. Since $\lim_{\mathrm{Re}\, \lambda \to \pm\infty} A(\lambda)/B(\lambda) = \prod_{j=1}^N \exp(\pm(\lambda_j - \alpha_j))$, this contour integral becomes $2\pi\mathrm{i}$ times the right-hand side of (43).

where $\sigma(\lambda) = \sigma(\lambda; 2\omega_1, 2\omega_3)$ is the Weierstrass σ function with periods $2\omega_1$ and $2\omega_3$. The symplectic form

$$\Omega = \sum_{j=1}^{N} \mathrm{d} \log B(\alpha_j) \wedge \mathrm{d}\alpha_j \tag{48}$$

is defined as in the rational and the trigonometric cases. Due to the oddness of the Weierstrass ζ function $\zeta(-\lambda) = -\zeta(\lambda)$, we have

$$\Omega = - \sum_{j,k=1}^{N} \zeta(\alpha_j - \lambda_k) \, \mathrm{d}\lambda_k \wedge \mathrm{d}\alpha_j$$

$$= - \sum_{j,k=1}^{N} \zeta(\lambda_k - \alpha_j) \, \mathrm{d}\alpha_j \wedge \mathrm{d}\lambda_k = \sum_{k=1}^{N} \mathrm{d} \log A(\lambda_k) \wedge \mathrm{d}\lambda_k. \tag{49}$$

We set $u_0 = \sum_{j=1}^{N} \alpha_j$ and regard u_0 and the coefficients in expansion of $A(\lambda)$,

$$A(\lambda) = \sum_{n=1}^{N} u_n f_n(\lambda; u_0), \tag{50}$$

as commuting Hamiltonians, where each $f_n(\lambda; u_0)$ has the same quasi-periodicity with $A(\lambda)$, namely,

$$f_n(\lambda + 2\omega_i; u_0) = (-1)^N \mathrm{e}^{2\eta_i(N\lambda - u_0)} f_n(\lambda; u_0), \quad \eta_i := \zeta(\omega_i) \tag{51}$$

for $i = 1, 3$. (The space of functions with quasi-periodicity (51) is N-dimensional.) For example, the functions

$$f_n(\lambda; u_0) = \sigma(\lambda)^N \phi_{u_0}^{(N-n)}(\lambda), \quad n = 1, \ldots, N, \tag{52}$$

fit our purpose, where $\phi_c(\lambda) = \sigma(\lambda - c)/\sigma(\lambda)\sigma(-c)$, but in the following we do not need the explicit form.

The coefficients $u_0, \ldots, u_N$ in the expansion (50) are functions of the variables α_j's and do not depend on $B(\lambda)$. Hence they commute with each other with respect to the Poisson bracket; $\{u_m, u_n\} = 0$.

But they *cannot* be independent because $A(\lambda)$ depends on N parameters $\{\alpha_j\}$. Regarding, for example, u_N as a function of independent parameters $u_0, \ldots, u_{N-1}$, we rewrite the symplectic form Ω as follows:

$$\Omega = \sum_{k=1}^{N} \mathrm{d} \log A(\lambda_k) \wedge \mathrm{d}\lambda_k = \sum_{k=1}^{N} \frac{\mathrm{d}A(\lambda_k)}{A(\lambda_k)} \wedge \mathrm{d}\lambda_k = \sum_{k=1}^{N} \sum_{n=0}^{N-1} \mathrm{d}u_n \wedge \frac{\partial A(\lambda_k)}{\partial u_n} \frac{\mathrm{d}\lambda_k}{A(\lambda_k)}$$

$$= \sum_{k=1}^{N} \mathrm{d}u_0 \wedge \left(\sum_{l=1}^{N} u_l \frac{\partial f_l}{\partial u_0}(\lambda_k; u_0) + \frac{\partial u_N}{\partial u_0} f_N(\lambda_k; u_0) \right) \frac{\mathrm{d}\lambda_k}{A(\lambda_k)} + \sum_{k=1}^{N} \sum_{n=1}^{N-1} \mathrm{d}u_n$$

$$\wedge \left(f_n(\lambda_k; u_0) + \frac{\partial u_N}{\partial u_n} f_N(\lambda_k; u_0) \right) \frac{\mathrm{d}\lambda_k}{A(\lambda_k)}. \tag{53}$$

Hence, introducing the generating function of the canonical transformation as the Abelian integral of the Seiberg–Witten differential $\log A(\lambda) \, d\lambda$ on the curve $C_u = \{(\lambda, z) | z = A(\lambda)\}$ (a subvariety of the total space of the line bundle over the elliptic curve defined by the quasi-periodicity (51)),

$$S(\lambda_0, \ldots, \lambda_{N-1}; u_0, \ldots, u_{N-1}) = \sum_{k=0}^{N} \int^{\lambda_k} \log A(\lambda) \, d\lambda, \tag{54}$$

we can express the "angle variable" ϕ_n by

$$\phi_0 = \frac{\partial S}{\partial u_0} = \sum_{k=1}^{N} \int^{\lambda_k} \left(\sum_{l=1}^{N} u_l \frac{\partial f_l}{\partial u_0}(\lambda; u_0) + \frac{\partial u_N}{\partial u_0} f_N(\lambda; u_0) \right) \frac{d\lambda}{A(\lambda)},$$

$$\phi_n = \frac{\partial S}{\partial u_n} = \sum_{k=1}^{N} \int^{\lambda_k} \left(f_n(\lambda; u_0) + \frac{\partial u_N}{\partial u_n} f_N(\lambda; u_0) \right) \frac{d\lambda}{A(\lambda)} \tag{55}$$

for $n = 1, \ldots, N - 1$. The canonical 2-form Ω is expressed as

$$\Omega = \sum_{n=0}^{N-1} du_n \wedge d\phi_n. \tag{56}$$

Hamiltonians u_n have similar expressions as the trigonometric case in terms of the coordinates λ_k, $z_k = A(\lambda_k)$. There are N equations to be satisfied

$$z_j = \sum_{n=1}^{N} u_n f_n(\lambda_j; u_0) \tag{57}$$

for $j = 1, \ldots, N$, or equivalently,

$$\begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_N \end{pmatrix} = \begin{pmatrix} f_1(\lambda_1; u_0) & f_2(\lambda_1; u_0) & \cdots & f_N(\lambda_1; u_0) \\ f_1(\lambda_2; u_0) & f_2(\lambda_2; u_0) & \cdots & f_N(\lambda_2; u_0) \\ \vdots & \vdots & \ddots & \vdots \\ f_1(\lambda_N; u_0) & f_2(\lambda_N; u_0) & \cdots & f_N(\lambda_N; u_0) \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \end{pmatrix}. \tag{58}$$

The determinant of the matrix in the right-hand side factors is as follows:

$$D(\lambda_1, \ldots, \lambda_N; u_0) := \begin{vmatrix} f_1(\lambda_1; u_0) & f_2(\lambda_1; u_0) & \cdots & f_N(\lambda_1; u_0) \\ f_1(\lambda_2; u_0) & f_2(\lambda_2; u_0) & \cdots & f_N(\lambda_2; u_0) \\ \vdots & \vdots & \ddots & \vdots \\ f_1(\lambda_N; u_0) & f_2(\lambda_N; u_0) & \cdots & f_N(\lambda_N; u_0) \end{vmatrix}$$

$$= (\text{non-zero constant}) \times \sigma \left(\sum_{i=1}^{N} \lambda_i - u_0 \right) \prod_{1 \le i < j \le N} \sigma(\lambda_i - \lambda_j). \tag{59}$$

The proof is given in Appendix A. Therefore Eq. (58) is generically non-degenerate and solved as

$$
\begin{aligned}
A(\lambda) &= u_1 f_1(\lambda; u_0) + u_2 f_2(\lambda; u_0) + \cdots + u_{N-1} f_{N-1}(\lambda; u_0) \\
&= D(\lambda_1, \ldots, \lambda_N; u_0)^{-1}
\begin{vmatrix}
0 & f_1(\lambda; u_0) & \cdots & f_N(\lambda; u_0) \\
-z_1 & f_1(\lambda_1; u_0) & \cdots & f_N(\lambda_1; u_0) \\
\vdots & \vdots & \ddots & \vdots \\
-z_N & f_1(\lambda_N; u_0) & \cdots & f_N(\lambda_N; u_0)
\end{vmatrix}.
\end{aligned}
\tag{60}
$$

In this formula u_0 has not yet been expressed as the function of λ_k and z_k. It is determined as an implicit function. Taking the logarithm of (46), we have

$$
\begin{aligned}
\log A(\mu) &= \sum_{j=1}^{N} \log \sigma(\mu - \alpha_j) = \sum_{\lambda = \text{root of } A(\lambda)} \log \sigma(\mu - \lambda) \\
&= \frac{1}{2\pi i} \oint \log \sigma(\mu - \lambda) \, d_\lambda \log A(\lambda),
\end{aligned}
$$

where the integration contour surrounds each α_j once but not $\mu + 2\omega_1 n_1 + 2\omega_3 n_3$ ($n_1, n_3 \in \mathbb{Z}$). Hence, e.g., fixing μ to 0, we have an equation

$$
\log A(0) = \frac{1}{2\pi i} \oint \log \sigma(-\lambda) \, d_\lambda \log A(\lambda).
\tag{61}
$$

Substituting (60) into this equation, we have an equation which fixes u_0.

6. Perspective from symplectic surfaces

Let us reconsider the integrable systems of the three types from the point of view of symplectic or Poisson surfaces [19–21].

The complex surface X for the rational case is essentially the (z, λ) plane with the line $z = 0$ deleted. The symplectic structure is defined by the 2-form

$$
\omega = \frac{dz \wedge d\lambda}{z}.
\tag{62}
$$

This surface is the affine part of a rational surface fibered over $\mathbb{P}^1$. dz/z is a holomorphic differential on the fibers $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ (i.e., cylinders). The symplectic structure on X induces a symplectic structure on the (smooth part of) $(N-1)$-fold symmetric product $X^{(N-1)} = X^{(N-1)}/S_{N-1}$ with the symplectic form

$$
\Omega = \sum_{j=1}^{N-1} \frac{dz_j \wedge d\lambda_j}{z_j},
\tag{63}
$$

where $(z_j, \lambda_j)_{j=1}^{N-1}$ is an $(N-1)$-tuple of points of X that represent a point of $X^{(N-1)}$. The interpolation formulae (21) and (23) imply that an $(N-1)$-tuple of points of X in *general*

position uniquely determines a curve of the form $C_u = \{(\lambda, z)|z = A(\lambda)\}$ that passes the $N - 1$ points. We thus obtain a mapping

$$(\lambda_j, z_j)_{j=1}^{N-1} \mapsto (u_2, \ldots, u_N) \tag{64}$$

from an open subset of $X^{(N-1)}$ to $\mathbb{C}^{N-1}$, the fibers of which are Lagrangian subvarieties with respect to Ω. This is a geometric interpretation of the integrable system on the moduli spaces of rational functions.

The same interpretation carries over to the trigonometric and elliptic cases. The symplectic surface X for these cases is also cylindrically fibered over a Riemann surface Σ (cylinder or torus), and the symplectic form can be written in the same form

$$\omega = \frac{\mathrm{d}z \wedge \mathrm{d}\lambda}{z}$$

except that λ is a coordinate on Σ.

We now turn to integrable systems associated with an elliptically fibered symplectic surface [22]. It should be noted that some part of the structure outlined in Section 2 is no longer retained or largely modified. In particular, there is no counterpart of the first system of canonical coordinates (α_j, ψ_j); it is the second and third systems of canonical coordinates (i.e., action–angle variables and "separated variables") that play a central role.

The first example is a specialization of Beauville's integrable systems [26] to a K3 surface X with elliptic fibration $X \to \mathbb{P}^1$. An affine model of this surface is defined by the equation

$$y^2 = 4z^3 + g_2(\lambda)z + g_3(\lambda) \tag{65}$$

of the Weierstrass normal form. $g_2(\lambda)$ and $g_3(\lambda)$ are (generic) polynomials of degrees 8 and 12, respectively. The complex symplectic structure is defined by the 2-form

$$\omega = \frac{\mathrm{d}z \wedge \mathrm{d}\lambda}{y}. \tag{66}$$

Note that the differential $\mathrm{d}z/z$ along the cylindrical fibers is now replaced by $\mathrm{d}z/y$ on the elliptic fibers. A canonically conjugate variable of λ is given by the elliptic integral (along the fibers of $X \to \mathbb{P}^1$)

$$\mu(z, \lambda) = \int_{(\infty,\infty)}^{(\lambda,z)} \frac{\mathrm{d}z}{y}, \tag{67}$$

the inversion of which is given by the Weierstrass $\wp$ function $z = \wp(\mu)$ with λ-dependent primitive periods.

The construction of the integrable system proceeds as follows [22]:

1. Choose a five-parameter family of curves C_u in X cut out by the equation

$$z = A(\lambda) = \sum_{n=1}^{5} u_n \lambda^{5-n}. \tag{68}$$

C_u is thus a hyperelliptic curve of genus 5 defined by the equation

$$y^2 = 4A(\lambda)^3 + g_2(\lambda)A(\lambda) + g_3(\lambda). \tag{69}$$

2. The phase space of the integrable system is (an open subset of) the fivefold symmetric product $X^{(5)}$ equipped with the symplectic form

$$\Omega = \sum_{j=1}^{5} \frac{\mathrm{d}z_j \wedge \mathrm{d}\lambda_j}{y_j}, \tag{70}$$

where the 5-tuple $(\lambda_j, y_j, z_j)_{j=1}^{5}$ of points of X represents a point of $X^{(5)}$. A set of canonical coordinates are given by λ_j and $\mu_j = \mu(A(\lambda_j), \lambda_j)$, $j = 1, \ldots, 5$. If λ_j are mutually distinct, the equations

$$z_j = A(\lambda_j) \tag{71}$$

can be solved for u_j, which are thus defined as functions on an open subset of $X^{(5)}$.

3. Plugging the equations $z_j = A(\lambda_j)$ into the expression of Ω and doing some algebra, one finds that

$$\Omega = \sum_{n=1}^{5} \mathrm{d}u_n \wedge \mathrm{d}\phi_n, \tag{72}$$

where ϕ_n are defined by the Abelian integrals

$$\phi_n = \sum_{j=1}^{5} \int_{(\infty,\infty)}^{(\lambda_j, y_j)} \frac{\lambda^{5-n}\,\mathrm{d}\lambda}{y}, \tag{73}$$

of the holomorphic differentials $\lambda^{5-n}\,\mathrm{d}\lambda/y$ on C_u. This expression of Ω shows that u_n and ϕ_n are action–angle variables. In particular, the fibers of the mapping $(\lambda_j, y_j, z_j)_{j=1}^{5} \mapsto (u_1, \ldots, u_5)$ on an open subset of $X^{(5)}$ turns out to be Lagrangian subvarieties.

The separated variables (λ_j, μ_j) and the action–angle variables (u_n, ϕ_n) are connected by a canonical transformations. The generating function takes the form

$$S = \sum_{j=1}^{5} \int_{\infty}^{\lambda_j} \mu(A(\lambda), \lambda)\,\mathrm{d}\lambda, \tag{74}$$

and the canonical transformation is defined by

$$\frac{\partial S}{\partial \lambda_j} = \mu_j, \qquad \frac{\partial S}{\partial u_n} = \phi_n. \tag{75}$$

Behind this construction is the Seiberg–Witten differential

$$\mathrm{d}S = \mu(A(\lambda), \lambda)\,\mathrm{d}\lambda. \tag{76}$$

The logarithmic factor $\log A(\lambda)$ in the cylindrically fibered case is thus replaced by the elliptic integral $\mu(A(\lambda), \lambda)$.

A variant of this integrable system is obtained from a rational surface with elliptic fibration [22]. Such a surface, too, can be defined by the Weierstrass normal form (65); $g_2(\lambda)$ and $g_3(\lambda)$ in this case are polynomials of degrees 4 and 6. The symplectic form ω (66) has poles along a compactification divisor at infinity. As a family of curves, we choose the

two-parameter family C_u cut out by the equation

$$z = A(\lambda) = c\lambda^2 + u_1\lambda + u_2. \tag{77}$$

C_u is a hyperelliptic curve of genus 2. Note that the setting is slightly different from the case of the elliptically fibered K3 surface—whereas u_1 and u_2 are moduli, c should be treated as a central element (Casimir function) of a Poisson structure. Apart from this difference, the construction of an integrable system is fully parallel. The phase space is realized as (an open subset of) the twofold symmetric product $X^{(2)}$; the Hamiltonians u_1 and u_2 are defined on an open subset of $X^{(2)}$ by the equations $z_j = A(\lambda_j)$; angle variables conjugate to (u_1, u_2) are given by the Abelian integrals

$$\phi_1 = \sum_{j=1,2} \int_{(\infty,\infty)}^{(\lambda_j, y_j)} \frac{\lambda\, d\lambda}{y}, \qquad \phi_2 = \sum_{j=1,2} \int_{(\infty,\infty)}^{(\lambda_j, y_j)} \frac{d\lambda}{y} \tag{78}$$

of holomorphic differentials on C_u; a generating function S for these action–angle variables can be defined in exactly the same way.

These two examples can be thought of as a generalization of the integrable system on the moduli space of rational functions. In particular, the Hamiltonians u_n are constructed by the same Lagrange interpolation formula, so that they take the same form (21) and (23), once written in the (local) coordinates (λ_k, z_k) on the symmetric product of X. One will be further tempted to increase the degrees of $A(\lambda)$, $g_2(\lambda)$ and $g_3(\lambda)$ to, say, $2m$, $4m$ and $6m$ ($m = 2, 3, \dots$). This is somewhat problematical. The genus $3m-1$ of the (still hyperelliptic) curve C_u then exceeds the number $2m + 1$ of the moduli of the polynomial $A(\lambda)$. One thus has to construct an integrable system from at most $(2m + 1)$-tuple of points on a family of curves of genus greater than $2m + 1$—a considerably unusual setting. Nevertheless, the construction seems to work at least formally.

One can conversely start from the systems associated with an elliptically fibered surface, and consider the system on the moduli space of rational functions as a kind of degeneration. From this point of view, we find some other types of degeneration in accordance with degeneration of the elliptic function $z = \wp(\mu)$ (and the associated singular rational curve), e.g.:

1. trigonometric (hyperbolic) function:

$$\mu = \int^z \frac{dz}{2\sqrt{z(z-1)}}, \quad z = \coth^2\mu; \tag{79}$$

2. quadratic function:

$$\mu = \int^z \frac{dz}{2\sqrt{z}}, \quad z = \mu^2; \tag{80}$$

3. exponential function:

$$\mu = \int^z \frac{dz}{z}, \quad z = e^\mu; \tag{81}$$

4. linear function:

$$\mu = \int^z dz, \quad z = \mu; \tag{82}$$

which are also known to emerge in a correspondence between the Painlevé equations and the (generalized) Calogero systems [28,29]. The third one in this list is nothing but the case of the moduli space of rational functions. The construction of an integrable system for the other cases is fully parallel to that case, except that the canonical coordinates (α_j, ψ_j) are no longer given by the zeros of $A(\lambda)$, etc.

We can further seek generalization to elliptic fibration over a Riemann surface Σ other than the sphere. Of particular interest is the case of an elliptically fibered surface over an elliptic curve. Recent work of Braden et al. [27] appears to provide a lot of material on this issue.

7. Conclusion

Starting from an integrable Hamiltonian system on the moduli space of rational functions, we constructed several integrable systems. As we have seen in Section 6, the phase spaces of them are the symmetric product $X^{(N)}$ of a complex symplectic surface X. The surface for the systems considered in Sections 3–5 are the fibration over Riemann surfaces (the Riemann sphere, a cylinder and a torus) with the cylinder as a fiber, while the systems reviewed in Section 6 are based on the symplectic surfaces elliptically fibered over $\mathbb{P}^1$.

These systems have other common features besides their algebro-geometric nature:

- They have two specific sets of canonical variables: the action–angle variables (u_n, ϕ_n) and the separated variables (λ_k, z_k).
- The generating function of the canonical transformation between (u_n, ϕ_n) and (λ_k, z_k) is the Abelian integral of the Seiberg–Witten differential on the spectral curve.
- The Hamiltonians $(u_1, \ldots, u_N)$ are explicitly described by the interpolation formula in terms of the separated variables (λ_k, z_k). The map $(\lambda_k, z_k) \mapsto (u_1, \ldots, u_N)$ gives a Lagrangian fibration of the phase space $X^{(N)}$.

Our construction is quite explicit and easy to generalize. One would have many other variants, using other symplectic surfaces, but the explicit description of the system would be more difficult.

Acknowledgements

Appendix A. Proof of (59)

In this appendix, we prove the formula (59).

Let us denote the space of meromorphic functions $f(\lambda)$ of $\lambda \in \mathbb{C}$ with the following properties by $\mathcal{L}$:

- the poles of which are located only on the lattice $2\omega_1\mathbb{Z} + 2\omega_3\mathbb{Z}$ and of order not greater than N.
- $f(\lambda + 2\omega_i) = e^{-2\eta_i u_0} f(\lambda)$ for $i = 1, 3$.

The linear space $\mathcal{L}$ is spanned by the functions,

$$\phi(\lambda), \phi'(\lambda), \ldots, \phi^{(N-1)}(\lambda), \tag{A.1}$$

where $\phi(\lambda) = \phi_{u_0}(\lambda)$ (see (52)). Assume that $\sum_{i=0}^{N-1} \lambda_i = u_0$ and set $f(\lambda) = \sigma(\lambda)^{-N} \prod_{i=0}^{N-1} \sigma(\lambda - \lambda_i)$, which belongs to $\mathcal{L}$. Expanding $f(\lambda)$ by the basis (A.1), we have a system of N linear equations

$$0 = a_{N-1}\phi(\lambda_j) + a_{N-2}\phi'(\lambda_j) + \cdots + a_0\phi^{(N-1)}(\lambda_j) \tag{A.2}$$

for $j = 0, \ldots, N - 1$, which has a non-trivial solution $(a_0, \ldots, a_{N-1})$. Therefore the function of λ defined by

$$
\begin{aligned}
d_N(\lambda) &= d_N(\lambda; \lambda_1, \ldots, \lambda_{N-1}) \\
&:= \begin{vmatrix}
\phi^{(N-1)}(\lambda) & \cdots & \phi'(\lambda) & \phi(\lambda) \\
\phi^{(N-1)}(\lambda_1) & \cdots & \phi'(\lambda_1) & \phi(\lambda_1) \\
\vdots & \ddots & \vdots & \vdots \\
\phi^{(N-1)}(\lambda_{N-1}) & \cdots & \phi'(\lambda_{N-1}) & \phi(\lambda_{N-1})
\end{vmatrix}
\end{aligned}
\tag{A.3}
$$

has a zero at $\lambda = \lambda_0 = c_A - \sum_{i=1}^{N-1} \lambda_i$. In addition, there are $N - 1$ trivial zeros of $d_N(\lambda)$ at $\lambda = \lambda_1, \ldots, \lambda_{N-1}$. Hence, from the periodicity with respect to the lattice $2\omega_1\mathbb{Z} + 2\omega_3\mathbb{Z}$ and the order of poles, it follows that:

$$
\begin{aligned}
&d_N(\lambda; \lambda_1, \ldots, \lambda_{N-1}) \\
&= \tilde{d}_N(\lambda_1, \ldots, \lambda_{N-1})\sigma(\lambda)^{-N}\sigma\left(\lambda + \sum_{i=1}^{N-1}\lambda_i - u_0\right)\prod_{j=1}^{N-1}\sigma(\lambda - \lambda_j),
\end{aligned}
\tag{A.4}
$$

where $\tilde{d}_N$ does not depend on λ. To determine the function $\tilde{d}_N$, we have to compare only the coefficient of λ^{-N} of the Laurent expansion of both sides at $\lambda = 0$, and by induction we have

$$
\begin{vmatrix}
\phi^{(N-1)}(\lambda_1) & \cdots & \phi'(\lambda_1) & \phi(\lambda_1) \\
\phi^{(N-1)}(\lambda_2) & \cdots & \phi'(\lambda_2) & \phi(\lambda_2) \\
\vdots & \ddots & \vdots & \vdots \\
\phi^{(N-1)}(\lambda_N) & \cdots & \phi'(\lambda_N) & \phi(\lambda_N)
\end{vmatrix}
$$
$$
= \sigma(u_0)^{-1}\left(\prod_{j=1}^{N-1} j!\right)\sigma\left(\sum_{i=1}^{N}\lambda_i - u_0\right)\prod_{i=1}^{N}\sigma(\lambda_i)^{-N}\prod_{1\leq i<j\leq N}\sigma(\lambda_i - \lambda_j),
\tag{A.5}
$$

from which follows (59) with f_n defined by (52). In general, the determinant differs from this case only by a non-zero constant factor.

A similar formula with the derivatives of Weierstrass' $\wp$ function instead of ϕ was found in the 19th century, see [30, p. 458].

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